

INTER-UNIVERSAL TEICHMÜLLER THEORY AS AN ANABELIAN GATEWAY TO DIOPHANTINE GEOMETRY AND ANALYTIC NUMBER THEORY

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<https://www.kurims.kyoto-u.ac.jp/~motizuki/IUT%20as%20an%20Anabelian%20Gateway.pdf>

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§1. Overview via a famous quote of Poincaré

(cf. [Alien]; [EssLgc], §1.5; [EssLgc], Examples 2.4.7, 2.4.8, 3.3.2; [ClsIUT], §4)

- In this talk, we give an *overview of various aspects of IUT*, many of which may be regarded as *striking examples* of the famous quote of Poincaré to the effect that

“mathematics is the art of giving the same name to different things”.

— which was apparently *originally motivated* by various mathematical observations on the part of Poincaré concerning certain remarkable similarities betw’n *transformation group symmetries* of modular functions such as *theta functions*, on the one hand, and *symmetry groups* of the *hyperbolic geometry* of the *upper half-plane*, on the other — all of which are *closely related to IUT!*

- Here, we note that there are *three ways* in which this quote of Poincaré is related to IUT:
 - the *original motivation* of Poincaré (mentioned above),
 - the key IUT notions of *coricity/multiradiality* (cf. §2, §3),
 - *new applications* of the *Galois-orbit version of IUT* (cf. §4).
- One important theme: it is possible to acquire a *survey-level* understanding of IUT using only a knowledge of such elementary topics as
 - the elem. notions of *rings/fields/groups/monoids* (cf. §2),
 - the elem. geom. of the *proj. line/Riemann sphere* (cf. §3).
- A more detailed exposition of IUT may be found in
 - the *survey texts* [Alien], [EssLgc], as well as in
 - the *videos/slides* available at the following URLs:

<https://www.kurims.kyoto-u.ac.jp/~motizuki/ExpHorizIUT21/WS3/ExpHorizIUT21-InvitationIUT-notes.html>

<https://www.kurims.kyoto-u.ac.jp/~motizuki/ExpHorizIUT21/WS4/ExpHorizIUT21-IUTSummit-notes.html>

§2. Galois groups as abstract groups: the example of the N -th power map

(cf. [EssLgc], Example 2.4.8; [EssLgc], §3.2, §3.8)

- Let R be an *integral domain* (e.g., $\mathbb{Z} \subseteq \mathbb{Q}$) equipped with the action of a *group* G , ($\mathbb{Z} \ni$) $N \geq 2$. For simplicity, assume that $N = 1 + \cdots + 1 \neq 0 \in R$; R has *no nontrivial N -th roots of unity*. Write $R^\triangleright \subseteq R$ for the *multiplicative monoid* $R \setminus \{0\}$. Then let us observe that the *N -th power map* on R^\triangleright determines an *isomorphism of multiplicative monoids* equipped with actions by G

$$G \curvearrowright R^\triangleright \xrightarrow{\sim} (R^\triangleright)^N (\subseteq R^\triangleright) \curvearrowright G$$

that does *not arise* from a *ring homomorphism*, i.e., it is clearly *not compatible* with *addition* (cf. our assumption on $N!$).

- Let $\dagger R, \ddagger R$ be *two distinct copies* of the integral domain R , equipped with respective actions by *two distinct copies* $\dagger G, \ddagger G$ of the group G . We shall use similar notation for objects with labels “ \dagger ”, “ \ddagger ” to the previously introduced notation. Then one may use the *isomorphism of multiplicative monoids* arising from the *N -th power map* discussed above to *glue* together

$$\dagger G \curvearrowright \dagger R \supseteq (\dagger R^\triangleright)^N \xleftarrow{\sim} \ddagger R^\triangleright \subseteq \ddagger R \curvearrowright \ddagger G$$

the *ring* $\dagger R$ to the *ring* $\ddagger R$ along the *multiplicative monoid* $(\dagger R^\times)^N \xleftarrow{\sim} \ddagger R^\times$. This gluing is *compatible* with the respective actions of $\dagger G, \ddagger G$ relative to the isomorphism $\dagger G \xrightarrow{\sim} \ddagger G$ given by forgetting the labels “ \dagger ”, “ \ddagger ”, but, since the N -th power map is *not compatible* with *addition* (!), this isomorphism $\dagger G \xrightarrow{\sim} \ddagger G$ may be regarded either as an isomorphism of (“*coric*”, i.e., *invariant* with respect to the N -th power map) *abstract groups* (cf. the notion of “*inter-universality*”, as discussed in [EssLgc], §3.2, §3.8!) or as an isomorphism of groups equipped with actions on certain *multiplicative monoids*, but *not* as an isomorphism of (“*Galois*” — cf. the *inner automorphism indeterminacies* of SGA1!) groups equipped with actions on *rings/fields*.

- The problem of describing (certain portions of the) ring structure of $\dagger R$ in terms of the ring structure of $\ddagger R$ — in a fashion that is compatible with the gluing and via a single algorithm that may be applied to the common (cf. logical AND \wedge !) glued data to reconstruct simultaneously (certain portions of) the ring structures of both $\dagger R$ and $\ddagger R$, up to suitable relatively mild indeterminacies (cf. the theory of crystals!) — seems (at first glance/in general) to be hopelessly intractable (cf. the case of \mathbb{Z})!

One well-known elementary example: when $N = p$, working modulo p (cf. indeterminacies, analogy with crystals!), where there is a common ring structure that is compatible with the p -th power map!

This is precisely what is achieved in IUT (cf. quote of Poincaré!) by means of the multiradial algorithm for the Θ -pilot via

- anabelian geometry (cf. the abstract groups $\dagger G, \ddagger G!$);
 - the p -adic/complex logarithm, theta functions;
 - Kummer theory, to relate Frob.-/étale-like versions of objects.
- Main point:

The multiplicative monoid and abstract group structures (but not the ring structures!) are common (cf. “logical AND \wedge !”) to \dagger, \ddagger .

On the other hand, once one deletes the labels “ \dagger ”, “ \ddagger ” to secure a “common R ”, one obtains a meaningless situation, where the common glued data may be related via “ \dagger ” OR “ \vee ” via “ \ddagger ” to the common R , but not simultaneously to both!

- When $R = \mathbb{Z}$ (or, in fact, more generally, the ring of integers “ \mathcal{O}_F ” of a number field F — cf. the multiplicative norm map $N_{F/\mathbb{Q}} : F \rightarrow \mathbb{Q}$), one may consider the “height”

$$\log(|x|) \in \mathbb{R}$$

for $0 \neq x \in \mathbb{Z}$. Then the N -th power map of (i), (ii) corresponds, after passing to heights, to multiplying real numbers by N ; the multiradial algorithm corresponds to saying that the height is unaffected (up to a mild error term!) by multiplication by N , hence that the height is bounded!

§3. Analogy with the projective line/Riemann sphere

(cf. [EssLgc], Example 2.4.7; [Alien], §3.1; [EssLgc], §3.5, §3.10)

- Let k be a field (in fact, could be taken to be an arbitrary ring), R a k -algebra. Denote units of a ring by a superscript “ \times ”. Write \mathbb{A}^1 for the affine line $\text{Spec}(k[T])$ over k , \mathbb{G}_m for the open subscheme $\text{Spec}(k[T, T^{-1}])$ of \mathbb{A}^1 obtained by removing the origin.

Recall that \mathbb{A}^1 is equipped with a well-known natural structure of ring scheme over k , while \mathbb{G}_m is equipped with a well-known natural structure of (multiplicative) group scheme over k . Moreover, we observe that the standard coordinate T on \mathbb{A}^1 and \mathbb{G}_m determines natural bijections:

$$\mathbb{A}^1(R) \xrightarrow{\sim} R, \quad \mathbb{G}_m(R) \xrightarrow{\sim} R^\times$$

- Write $\dagger\mathbb{A}^1, \ddagger\mathbb{A}^1$ for the k -ring schemes given by copies of \mathbb{A}^1 equipped with labels “ \dagger ”, “ \ddagger ”. Observe that there exists a unique isomorphism of k -ring schemes $\dagger\mathbb{A}^1 \xrightarrow{\sim} \ddagger\mathbb{A}^1$; moreover, there exists a unique isomorphism of k -group schemes

$$(-)^{-1} : \dagger\mathbb{G}_m \xrightarrow{\sim} \ddagger\mathbb{G}_m$$

that maps $\dagger T \mapsto \ddagger T^{-1}$. Note that $(-)^{-1}$ does not extend to an isomorphism $\dagger\mathbb{A}^1 \xrightarrow{\sim} \ddagger\mathbb{A}^1$ and is clearly not compatible with the k -ring scheme structures of $\dagger\mathbb{A}^1 (\supseteq \dagger\mathbb{G}_m), \ddagger\mathbb{A}^1 (\supseteq \ddagger\mathbb{G}_m)$.

- The *standard construction* of the *projective line* \mathbb{P}^1 may be understood as the result of *gluing* $\dagger\mathbb{A}^1$ to $\ddagger\mathbb{A}^1$ along the isomorphism

$$\dagger\mathbb{A}^1 \supseteq \dagger\mathbb{G}_m \xrightarrow{(-)^{-1}} \ddagger\mathbb{G}_m \subseteq \ddagger\mathbb{A}^1$$

— i.e., at the level of *R-rational points*

$$\dagger R \supseteq \dagger R^\times \xrightarrow{(-)^{-1}} \ddagger R^\times \subseteq \ddagger R$$

— where $\square R \stackrel{\text{def}}{=} \square\mathbb{A}^1(R)$, $\square R^\times \stackrel{\text{def}}{=} \square\mathbb{G}_m(R)$, for $\square \in \{\dagger, \ddagger\}$ (cf. the *gluing* situation discussed in §2, where “ $(-)^{-1}$ ” corresponds to “ $(-)^N$!”). Thus, *relative to this gluing*, we observe that there exists a *single rational function* on the copy of “ \mathbb{G}_m ” that appears in the gluing that is *simultaneously* equal to the rational function $\dagger T$ on $\dagger\mathbb{A}^1$ AND [cf. “ \wedge ”!] to the rational function $\ddagger T^{-1}$ on $\ddagger\mathbb{A}^1$.

- *Summary:*

The standard construction of the *projective line* may be regarded as consisting of a *gluing* of two *ring schemes* along an *isomorphism* of *multiplicative group schemes* that is *not compatible* with the *ring scheme* structures on either side of the gluing.

Finally, we observe that if, in the gluing under discussion, one *arbitrarily deletes* the *distinct labels* “ \dagger ”, “ \ddagger ” (e.g., on the grounds that both ring schemes represent “THE” structure sheaf “ \mathcal{O}_X ” of a k -scheme X !), then the resulting “*gluing without labels*” amounts to a gluing of a *single copy* of \mathbb{A}^1 to itself that maps the standard coordinate T on \mathbb{A}^1 (regarded, say, as a rational function on \mathbb{A}^1) to T^{-1} . That is to say, such a *deletion of labels* (even when restricted to the (abstractly isomorphic) multiplicative monoids $\dagger T^\mathbb{Z}$, $\ddagger T^\mathbb{Z}$!) immediately results in a *contradiction* (i.e., since $T \neq T^{-1}$!), unless one passes to some sort of *quotient* of \mathbb{A}^1 . On the other hand, passing to such a quotient amounts, from a foundational/logical point of view, to the introduction of some sort of *indeterminacy*, i.e., to the consideration of some sort of *collection of possibilities* [cf. “ \vee ”!].

- When $k = \mathbb{C}$ (i.e., the complex number field), one may think of the projective line \mathbb{P}^1 as the Riemann sphere \mathbb{S}^2 equipped with the Fubini-Study metric and of the gluing under discussion as the gluing, along the equator \mathbb{E} , of the northern hemisphere \mathbb{H}^+ to the southern hemisphere \mathbb{H}^- . Then the discussion above of the standard coordinates “ $\dagger T$ ”, “ $\ddagger T$ ” translates into the following (at first glance, self-contradictory!) phenomenon:

an oriented flow along the equator — which may be thought of physically as a sort of east-to-west wind current — appears simultaneously to be flowing in the clockwise direction, from the point of view of $\mathbb{H}^+ \subseteq \mathbb{S}^2$, AND in the counterclockwise direction, from the point of view of $\mathbb{H}^- \subseteq \mathbb{S}^2$.

In particular, if one arbitrarily deletes the labels “+”, “-” and identifies \mathbb{H}^- with \mathbb{H}^+ , then one does indeed literally obtain a contradiction. On the other hand, one may relate \mathbb{H}^- to \mathbb{H}^+ (not by such an arbitrary deletion of labels (!), but rather) by applying

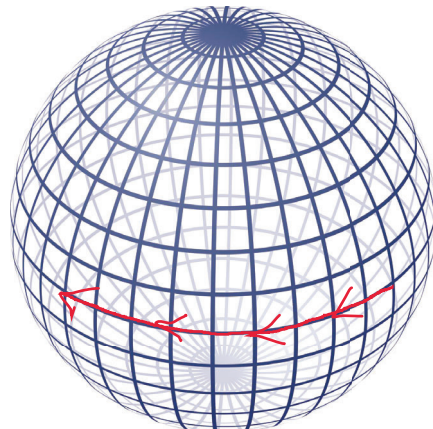
the metric/geodesic geometry of \mathbb{S}^2 — i.e., by considering the geodesic flow along great circles/lines of longitude — to represent, up to a relatively mild distortion, the entirety of \mathbb{S}^2 , i.e., including $\mathbb{H}^- \subseteq \mathbb{S}^2$, as a sort of deformation/displacement of \mathbb{H}^+ (cf. the point of view of cartography!).

It is precisely this metric/geodesic approach that corresponds to the anabelian geom.-based multiradial algorithm for the Θ -pilot in IUT (cf. the analogy discussed in [Alien], §3.1, (iv), (v), as well as in [EssLgc], §3.5, §3.10, between multiradiality and connections/parallel transport/crystals!).

northern hemisphere \mathbb{H}^+

———— equator \mathbb{E} —————

southern hemisphere \mathbb{H}^-



§4. Brief preview of the Galois-orbit version of IUT

(cf. “Expanding Horizons” *videos/slides* cited in §1)

- New applications (work in progress!) of
 Galois-orbit version of IUT (GalOrbIUT):
 - the (local-global) *Section Conjecture*, i.e., the *geometricity* of any Galois section (of the arithmetic fundamental group of a hyperbolic curve s.t. ...) over a number field that is assumed to be geometric at every nonarchimedean prime;
 - *nonexistence of Siegel zeroes* of Dirichlet L -functions associated to imaginary quadratic number fields (i.e., by applying the work of Colmez/Granville-Stark/Táfula);
 - *numerically stronger* version of *abc/Szpiro* inequalities.
- That is to say, we obtain three *a priori different* applications to
 - *anabelian geometry* (local-global Section Conjecture),
 - *analytic number theory* (nonexistence of Siegel zeroes),
 - *Diophantine geometry* (abc/Szpiro inequalities)
 — a *striking example* of *Poincaré’s quote*, i.e., all three are essentially the *same mathematical phenomenon* of *bounding heights*, i.e., *bounding “local denominators”!*
- Here, the *local-global Section Conjecture* application is also noteworthy in that
 - it exhibits IUT as “*anabelian geometry* applied to obtain more *anabelian geometry!*” (less psychologically/intuitively surprising than the other two applications!);
 - it is *technically the most difficult/essential* (!) of the three, i.e., to the extent that the *other two* applications may be thought of, to a substantial extent, as being *“inessential by-products”*;
 - the *historical point of view* (cf., e.g., of Grothendieck’s famous “letter to Faltings”) suggests (*without any proof!*) that the Section Conjecture might imply results in Diophantine geometry (such as the Mordell Conjecture).

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